

The covering number $C(13, 6, 3)$ is 21

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Mathematical formalisation, proof-campaign design, implementation, hostile review, and manuscript development with Solly*.

Abstract

We prove by exhaustive, certificate-backed computation that the covering number $C(13, 6, 3)$ is 21. The previously recorded bounds were $20 \leq C(13, 6, 3) \leq 21$. For the lower bound, suppose that a 20-block cover exists. Every point link is a $(12, 5, 2)$ cover. Exhaustive generation gives no eight-block link and 107 nine-block links up to isomorphism. Incidence convexity then forces two degree-nine points of codegree four or five. We enumerate all compatible pairs of such links, quotient 632,792 parent-relative cores to 193,155 two-star classes, and quotient their remaining obligations to 188,540 residual shapes. When one residual shape occurs with more than one completion budget, the proof retains the maximum represented budget. Nonnegative-integer dual certificates exclude 182,542 residuals directly; exact include/exclude trees, whose 17,113 leaves are closed by the same dual inequality, exclude the remaining 5,998. For the upper bound, an explicit 21-block family covers all 286 triples. The full replay regenerates the finite frontiers and checks every certificate using integer arithmetic. No timeout, floating-point optimum, or generic solver UNSAT status is a proof premise.

1 What is proved

Choose 13 points. We seek the smallest collection of six-point groups such that every possible choice of three points appears together in at least one group. Twenty-one groups are sufficient, and this paper proves that twenty groups can never suffice.

The lower-bound proof has three stages:

1. counting forces a pair of points with a tightly constrained local structure;
2. exhaustive canonical generation lists every residual covering problem that can remain around such a pair; and
3. deterministically replayable integer certificates exclude every residual.

The claim has a direct falsifier. A valid 20-block cover disproves the lower bound. Equivalently, one generated residual class admitting a completion within its recorded six- or seven-block budget breaks the certificate chain.

*Solly is the working name for the collaborating AI instances used across OpenAI ChatGPT and Codex in this research programme; Codex provided implementation and tooling support.

2 Definitions and recorded boundary

A (v, k, t) covering design is a family of k -subsets, called blocks, of a v -element point set such that every t -subset is contained in at least one block. The covering number $C(v, k, t)$ is the minimum possible number of blocks. The standard recursive lower bound is due to Schönheim [4]. The frozen La Jolla Coverings Repository records constructive upper bounds and known exact values [1, 2]. In version 1.2 it records

$$20 \leq C(13, 6, 3) \leq 21.$$

The 21-block construction was known. The unresolved step was exclusion of a 20-block cover.

Wilder independently developed the value $C(12, 5, 2) = 9$, the resulting degree-nine and slack-three structure for a hypothetical 20-cover, and the 107-link catalogue [5]. We make no novelty claim for those ingredients or for Lemmas 1–2 below. The present contribution is the complete two-root quotient and exact exclusion of every residual completion.

Let $V = \{1, \dots, 13\}$ and let $\mathcal{C}$ be a family of six-subsets of V . For a point x , write

$$d_x = |\{B \in \mathcal{C} : x \in B\}|.$$

For distinct points x, y , write

$$\lambda_{xy} = |\{B \in \mathcal{C} : \{x, y\} \subseteq B\}|.$$

The link at x is obtained by removing x from every block containing it. If $\mathcal{C}$ covers every triple of V , then the link at x is a $(12, 5, 2)$ cover with d_x blocks.

3 The forced degree-nine pair

Lemma 1. Every point in a 20-block $(13, 6, 3)$ cover has degree at least nine.

Proof. Complete generation with nauty’s **genbg** gives no eight-block $(12, 5, 2)$ cover and exactly 107 nine-block covers up to isomorphism. Hence $d_x \geq 9$ for every point x . The nine-block command is

```
genbg -cq1 -d5:1 -D5:9 -Y1 9 12 45:45
```

Here the two bipartition cells represent nine candidate blocks and twelve points; the degree and edge constraints encode nine five-subsets covering all 66 point pairs. The corresponding eight-block generation is empty. The replay checks each emitted incidence graph directly, so the command supplies an exact finite enumeration rather than a heuristic search.

Lemma 2. A hypothetical 20-block cover has two degree-nine points with codegree at least four.

Proof. The sum of all point degrees is $20 \cdot 6 = 120$. Lemma 1 accounts for $13 \cdot 9 = 117$ incidences, leaving three surplus incidences. Thus the number m of degree-nine points is 10, 11, or 12.

Let S be those m points and put $r_B = |B \cap S|$. Then

$$\sum_{B \in \mathcal{C}} r_B = 9m, \quad \sum_{\{x, y\} \in \binom{S}{2}} \lambda_{xy} = \sum_{B \in \mathcal{C}} \binom{r_B}{2}.$$

Convexity of $\binom{r}{2}$ makes the right side smallest when the $9m$ incidences are distributed as evenly as possible among the 20 blocks:

m	convex floor	ceiling if every $\lambda_{xy} \leq 3$
10	160	135
11	196	165
12	240	198

Every floor exceeds its corresponding ceiling, so some pair of degree-nine points has codegree at least four.

Lemma 3. The forced pair has codegree four or five.

Proof. In the complete catalogue of 107 nine-block $(12, 5, 2)$ links, every point has degree three, four, or five. A point's degree inside the link at x is its codegree with x in the original cover. Lemma 2 therefore forces a pair x, y with $d_x = d_y = 9$ and $\lambda_{xy} \in \{4, 5\}$.

Rooting each link at each of its 12 points gives 1,284 assignments. Here a rooted class is the coloured incidence type of the blocks common to the first root and the proposed second root, not the isomorphism type of the complete rooted link. This is sufficient because the compatible second links depend only on that common-block configuration. Canonical isomorphism reduces the assignments to 67 classes: one of codegree three, 30 of codegree four, and 36 of codegree five. The unique codegree-three class cannot carry the forced pair and is discarded. The remaining 66 classes contain 899 rooted assignments.

4 Exhausting the two-star frontier

Fix the forced pair x, y and write $c = \lambda_{xy}$. The nine blocks through x and the nine through y have a union of

$$9 + 9 - c = 18 - c$$

fixed blocks. A 20-block cover therefore has exactly $c + 2$ blocks left: six when $c = 4$, and seven when $c = 5$. Every remaining block avoids both roots.

For each of the 66 load-bearing rooted link classes, the verifier regenerates every compatible second degree-nine link while preserving the common blocks. It selects matching common-block subsets from each of the 107 complete link types, enumerates every colour-preserving incidence isomorphism, transfers the remaining blocks, checks the resulting nine-block pair cover directly, and then quotients by the stabiliser of the first rooted link. This gives 632,792 parent-relative two-star cores. Canonical coloured-incidence labelling under every point relabelling, including exchange of x and y , reduces the frontier to 193,155 two-star isomorphism classes: 193,000 of codegree four and 155 of codegree five. Canonical labelling is performed with nauty [3].

Let $W = V \setminus \{x, y\}$, so $|W| = 11$. For each two-star core, retain only the triples of W not covered by a fixed block. These triples form a residual 3-uniform hypergraph. A completion block is one of the $\binom{11}{6} = 462$ six-subsets of W .

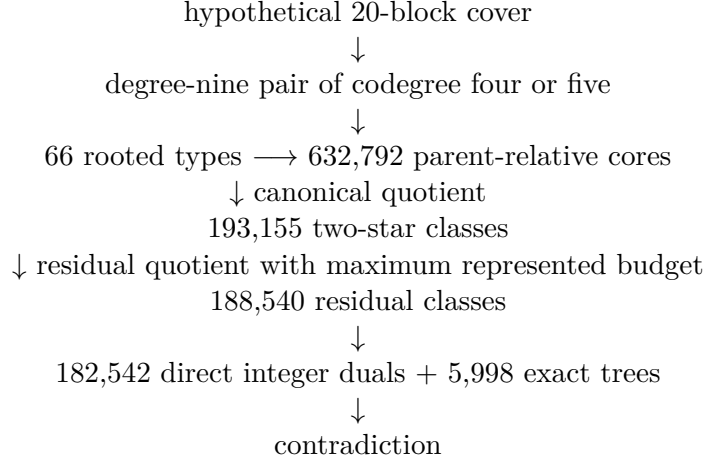
Canonical labelling reduces the 193,155 cores to 188,540 residual hypergraph shapes. Residual shape alone does not determine whether its parent left six or seven completion blocks. The corrected quotient therefore records every represented budget and binds the class to

$$b(H) = \max\{\text{remaining-block budget of a parent represented by } H\}.$$

This is conservative: excluding a completion at the largest represented budget excludes every smaller represented budget. Four shapes occur under both budgets. After retaining their maxima, 188,387 classes have budget six and 153 have budget seven.

No other information from a two-star parent is needed for this completion question. Once the two degree-nine links are fixed, every remaining block avoids both roots and is simply a six-subset of W . Its complete effect is therefore the set of residual triples it covers. Consequently, completion existence is invariant under isomorphism of the residual hypergraph together with its retained block budget.

Proof flow.



Stage	Count
nine-block (12, 5, 2) link orbits	107
rooted assignments	1,284
rooted isomorphism classes	67
load-bearing rooted classes	66
load-bearing rooted assignments	899
parent-relative two-star cores	632,792
two-star isomorphism classes	193,155
residual hypergraph shapes	188,540

Every quotient is regenerated during replay. The verifier assigns each object a nauty canonical coloured-incidence certificate and retains one representative per certificate. A separate canonical-labelling pass emits the point relabelling for every member. Python composes an explicit member-to-representative permutation and checks that it maps the member's complete block set, or residual hyperedge set, exactly onto the retained representative. Thus nauty proposes the grouping, but certificate-string equality alone cannot justify a merge. The verifier also checks that represented counts sum to the preceding frontier and that each residual is bound to the maximum budget among all represented parents.

5 Exact residual certificates

For a residual class, let E be its uncovered triples, let $\mathcal{A}$ be the 462 candidate completion blocks, and let b be its retained maximum completion budget.

Proposition 4 (integer dual exclusion). Suppose nonnegative integer weights w_e for $e \in E$, not all zero, and a positive integer D satisfy

$$\sum_{e \subseteq A} w_e \leq D \quad \text{for every } A \in \mathcal{A}, \quad \sum_{e \in E} w_e > bD.$$

Then no b candidate blocks cover E .

Proof. If b blocks covered E , charge each residual triple to one chosen block that covers it. The total charged weight would be $\sum_E w_e$, while each chosen block could receive weight at most D . Hence $\sum_E w_e \leq bD$, a contradiction.

The verifier checks this inequality using integer arithmetic for all 182,542 residual classes closed directly by a dual certificate. Some margins are deliberately thin: one valid certificate has total weight 600,000,417 against the strict threshold $6 \cdot 10^8$. Exact integers, rather than rounded floating-point comparisons, preserve that distinction. The remaining 5,998 residuals are closed by exact include/exclude trees over the 462 lexicographically indexed candidate blocks. Each internal record names one candidate and contains both include and exclude children. The include child removes the covered triples and reduces the budget by one; the exclude child removes only that candidate. Every one of the 17,113 leaves stores an integer-dual denominator and its positive obligation weights and is closed by the same proposition. The replay checks 28,228 tree nodes and has no heuristic or unproved structural leaves.

Consequently no residual class admits a completion within its retained maximum budget. By the preceding lemmas and the exhaustive quotient chain, no 20-block $(13, 6, 3)$ cover exists.

6 A 21-block cover

The following family covers all $\binom{13}{3} = 286$ triples:

$$\begin{array}{lll} \{1, 2, 3, 4, 5, 6\} & \{1, 2, 6, 8, 9, 12\} & \{1, 2, 6, 10, 11, 13\} \\ \{1, 2, 7, 8, 10, 11\} & \{1, 2, 7, 10, 11, 13\} & \{1, 3, 4, 7, 8, 12\} \\ \{1, 3, 4, 7, 9, 11\} & \{1, 3, 4, 10, 12, 13\} & \{1, 5, 6, 7, 11, 12\} \\ \{1, 5, 8, 9, 10, 13\} & \{2, 3, 5, 7, 10, 12\} & \{2, 3, 5, 8, 10, 12\} \\ \{2, 3, 5, 9, 11, 13\} & \{2, 4, 6, 7, 9, 10\} & \{2, 4, 8, 11, 12, 13\} \\ \{3, 4, 7, 8, 9, 11\} & \{3, 5, 6, 7, 8, 13\} & \{3, 6, 9, 10, 11, 12\} \\ \{4, 5, 6, 8, 10, 11\} & \{4, 5, 6, 9, 12, 13\} & \{4, 5, 7, 9, 12, 13\} \end{array}$$

Direct enumeration verifies 286 covered triples and zero uncovered triples. The construction is the random-greedy cover archived in the La Jolla Coverings Repository, dated 31 January 1997 [1].

7 Main theorem

Theorem 5. $C(13, 6, 3) = 21$.

Proof. The exact residual campaign excludes every 20-block cover. Any smaller cover could be extended to 20 blocks by adding blocks, so $C(13, 6, 3) \geq 21$. The explicit family above is a 21-block cover, so $C(13, 6, 3) \leq 21$.

8 Reproducibility and trust boundary

The distribution archive replays the complete load-bearing chain:

1. verify every distributed file against its checksum manifest;
2. compile the coloured-incidence canonicaliser against nauty;
3. regenerate the empty eight-block and 107-orbit nine-block catalogues;
4. regenerate all 66 load-bearing rooted second-link refinements;
5. rebuild both canonical quotients while retaining each residual’s maximum represented budget;
6. replay every integer dual and branch-and-dual tree; and
7. check the explicit 21-block cover against all 286 triples.

The tested environment was macOS x86_64, Python 3.11.3, NetworkX 3.5, NumPy 1.24.3, and nauty 2.9.3. The soundness boundary comprises the focused Python verifier, NetworkX graph isomorphism, nauty’s `genbg` enumeration, the C compiler, and integer arithmetic. Canonical labels are candidate grouping keys only: every proposed quotient merge is separately checked by an explicit point permutation, so canonical labelling is not a soundness premise. No timeout, floating-point optimum, heuristic geometry, or ordinary SAT/SCIP UNSAT status is accepted as a proof premise. An independent frontier or exact-cover encoding, or proof-assistant formalisation, remains desirable. Recent exact covering-number work using separately checked SAT certificates provides a useful methodological comparison [6].

Proof identity. The distributed theorem manifest `results/c1363-exact-result.json` has SHA-256

`e17d927ce4ffcca5582097605251b43bab305996d94861a395e1eeebc0602052`.

The proof archive, verification instructions, checksums, and public review record are indexed at <https://moletariat.ai/releases/c1363>. The outer archive SHA-256 is recorded on that release page and in the custody ledger rather than embedded in the archive it identifies, which would create a circular checksum.

Licensing. Verifier source and scripts are MIT licensed. Certificates, proof data, and documentation are CC BY 4.0. The archived La Jolla witness retains its source attribution and CC BY 4.0 terms.

Author contributions and AI disclosure

Ben Little directed the research programme, selected the problem, evaluated the claim boundary, and supervised release. Solly is the working name for the collaborating AI instances used across OpenAI ChatGPT and Codex in this research programme. Solly designed and implemented the finite proof campaign, certificate checks, hostile review, manuscript structure, and reproducibility record in collaboration with Ben Little; Codex supplied implementation and tooling support. All theorem statements, counts, hashes, and the explicit witness are subject to deterministic replay. AI participation does not substitute for those checks or for independent peer review.

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